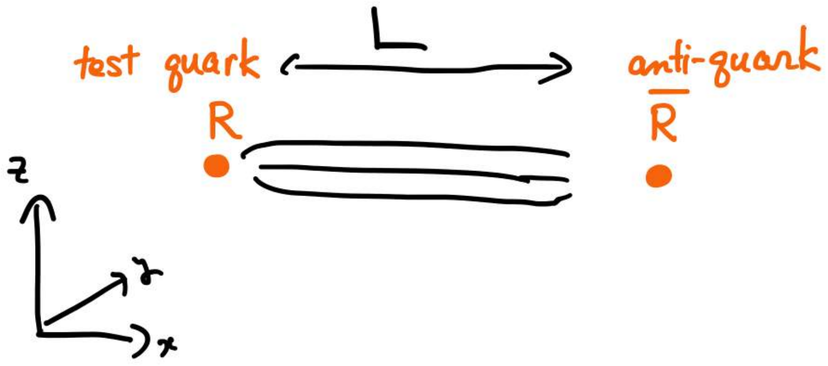
# Wilson-'t Hooft classification & $\mathcal{N}=1^*$ SYM gapped vacua from the viewpoint of 1-form symmetry

Yuya Tanizaki (Yukawa institute, Kyoto)

based on 2306.02485 & w.i.p. w/ Mendel Nguyen, Mithat Ünsal (NCSU)

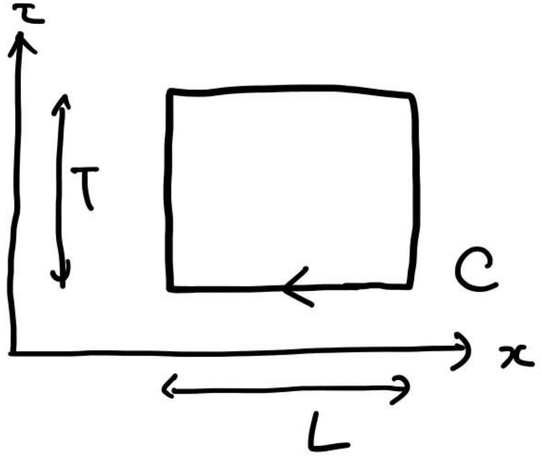
# Confinement




Interquark potential :

$$V_{R\bar{R}}(L) \simeq \begin{cases} \sigma_R L & \text{( confinement )} \\ \text{const.} & \text{( Higgs )} \end{cases}$$

Wilson loop : $W_R(C) = \mathrm{tr}_R \left[ \mathcal{P} \exp\left( \oint_C a \right) \right]$.




$$\langle W_R(C) \rangle \simeq \exp\left( - T \cdot V_{R\bar{R}}(L) \right)$$

$$\simeq \begin{cases} \exp(-\sigma_R T L) & \text{(Area law)} \\ \exp(-\text{const.}\,(T+L)) & \text{(Perimeter law)}. \end{cases}$$

This is an order parameter for the $\mathbb{Z}_N$ 1-form symmetry, or center symmetry.

't Hooft pointed out the importance of magnetic lines :

$$\begin{cases} W(C) : \text{worldline of test electric charge} \quad (R = \text{fund.}) \\ H(C,\Sigma) : \text{worldline of test magnetic charge} \quad (\text{'t Hooft loop}) \\ \qquad (\text{\& worldsheet of associated Dirac string}) \end{cases}$$

To see the usefulness of having both $W$ & $H$, let's see some examples for $SU(2)$ :

| | $W(C)$ | $H(C,\Sigma)$ | $W(C)H(C,\Sigma)$ |
|---|---|---|---|
| Higgs | $\boxed{\text{Perimeter}}$ | Area | Area |
| Confinement by monopoles | Area | $\boxed{\text{Perimeter}}$ | Area |
| Confinement by dyons | Area | Area | $\boxed{\text{Perimeter}}$. |

$\Rightarrow$ We get the fine classification for gapped vacua,

by specifying the charge $W_{\underbrace{(m,n)}_{\in\, \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N}}(C,\Sigma) = H(C,\Sigma)^m \, W(C)^n$ obeying the $\boxed{\text{perimeter law}}$.

# Wilson–'t Hooft commutation relation

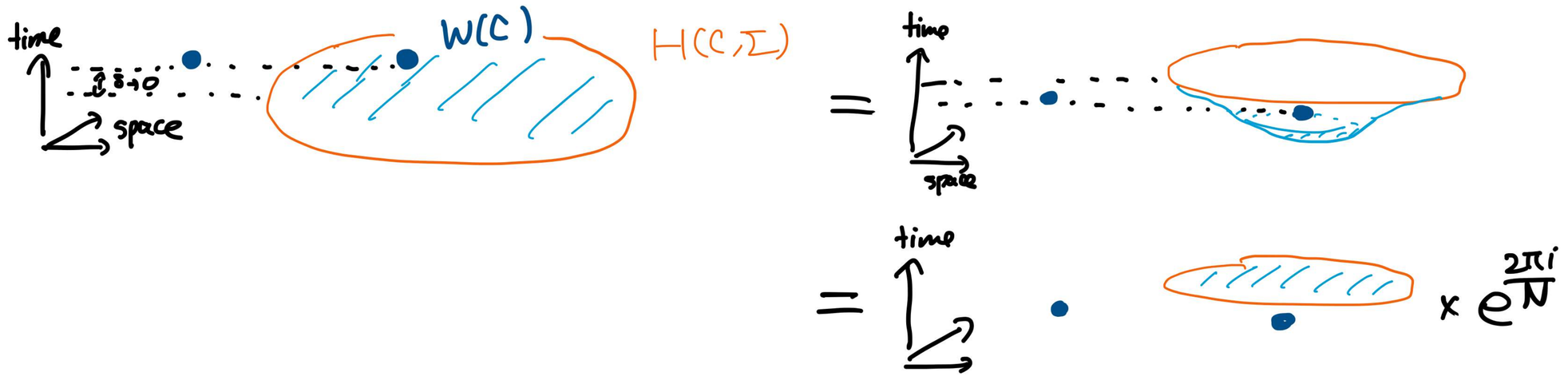
For Lie(SU(N)) – gauge theories only with adj. matters,

$$\hat{W}(C)\,\hat{H}(C') = \hat{H}(C')\,\hat{W}(C)\,e^{\frac{2\pi i}{N}\mathrm{Link}(C,C')}.$$

However, this violates locality of 4d QFTs if both $W$ & $H$ are lines.

- 't Hooft : This implies the formation of the confining string at least one of non-mutually local lines. ('t Hooft '78, '79, '81)

- Modern perspective : 't Hoof loop is not a genuine line.




Magnetic loop lives on the bdry of 1-form symmetry generator. (Aharony, Seiberg, Tachikawa '13)

# Is the Wilson-'t Hooft classification proven?

In the modern viewpoint, the non-commutativity of W & H does <u>not</u> (directly) indicate the presence of confining strings.

⇒ This was the key assumption in the conventional derivation for the W-'t H classification.

We give another derivation of the W-'tH classification based on the 1-form symmetry. (Nguyen, YI, Ünsal, 2306.02485)

# Main result

We show that

$$\mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} \mathbb{Z}_{\underline{\underline{n}}}^{[1]} \quad \text{enriched with the level-}k\ \mathbb{Z}_{\underline{\underline{n}}}^{[1]}\text{-SPT}$$

$$\overset{1:1}{\Longleftrightarrow} \quad \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N \longrightarrow \underline{H = \{x(\underline{\underline{n}},0) + y(k, -\tfrac{N}{\underline{\underline{n}}})\}} \quad \text{in Wilson-'t Hooft classification.}$$

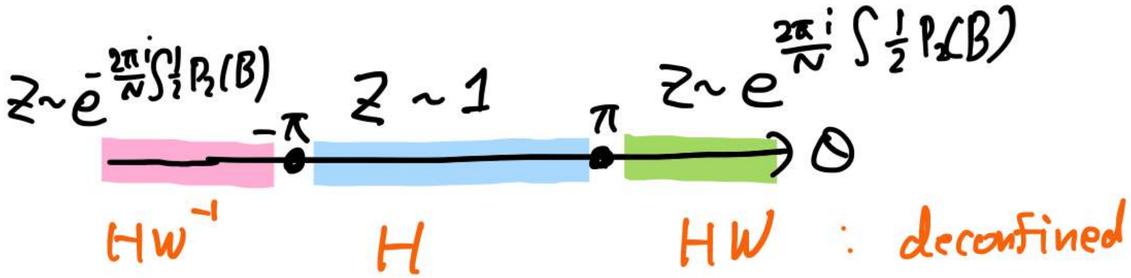
↖ Any order-$N$ subgroup of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ can be expressed in this way.

We can use the screening behavior of <u>non-genuine</u> dyonic lines to specify SPT.

(← Reminiscent of the Kennedy-Tasaki transformation of the Haldane gapped state)

<u>example</u> 4d pure YM + $\theta$-term

$W$ : always confined (at least in large-$N$)




(Gaiotto et al. 2017)

To get the information of symmetry, it's convenient to consider

$Z[B]$ : partition function w/ background gauge field $B$.

- Spontaneous symmetry breaking $\mathbb{Z}_N^{(1)} \xrightarrow{SSB} \mathbb{Z}_n^{(1)}$

$$Z[B] = \underbrace{\delta(nB)}_{\int_\Sigma B \text{ has to be quantized in } \frac{2\pi}{n}.} \times \cdots$$

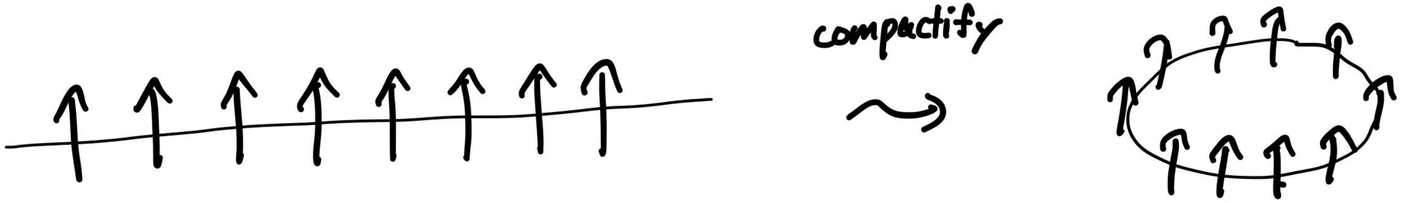
- Symmetry protected topological (SPT) state of $\mathbb{Z}_N^{(1)}$

$$Z[B] = e^{i\frac{Nk}{4\pi}\int B\wedge B} \qquad (k \sim k+N).$$

local topological action of $B$.

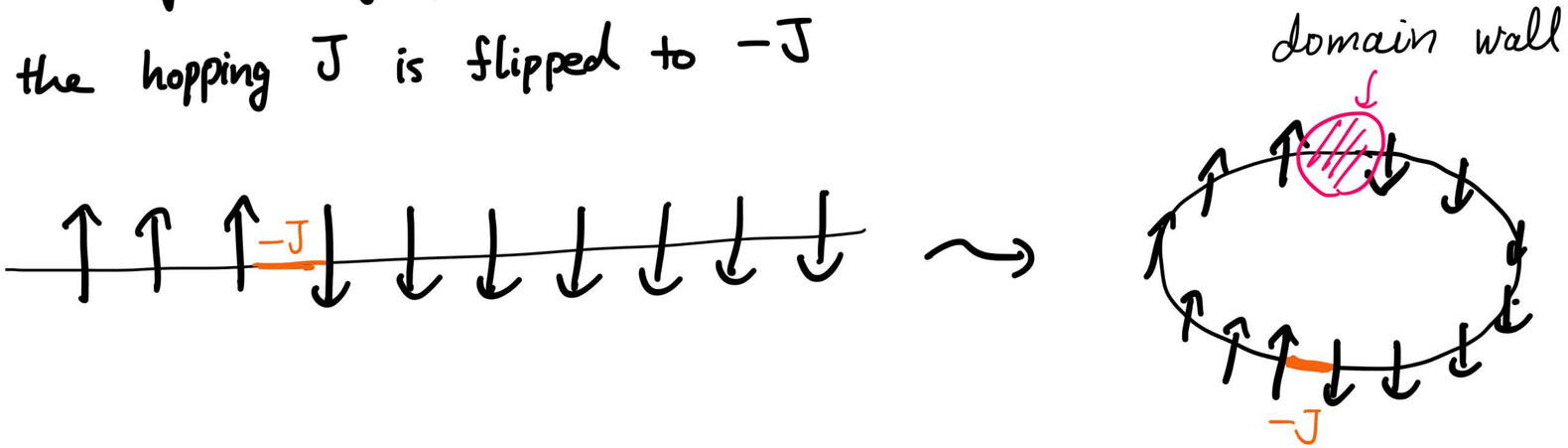
# SSB & background gauge field

Let's consider $\mathbb{Z}_2$ Ising spin for simplicity, and assume $\mathbb{Z}_2$ is spontaneously broken:



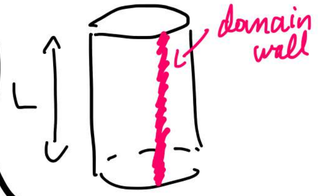

Introduce the background gauge field $A$ w/ $\int A = \pi$.

$\Leftrightarrow$ One of the hopping $J$ is flipped to $-J$




In the spacetime picture

$$Z[A] = \left\{ \begin{array}{ll} 1 & (A=0) \\ & \\ & (\int A = \pi) \; = e^{-L\cdot(\text{domain-wall tension})} \to 0 \end{array} \right\} = \delta[A].$$



With this knowledge, we can write down the low-energy effective action for

$\mathbb{Z}_N^{(1)} \xrightarrow{SSB} \mathbb{Z}_n^{(1)}$ w/ level-$k$ $\mathbb{Z}_n^{(1)}$-SPT stacking : (Nguyen, YT, Ünsal 2306. see also Damia, et al. 2305.17084)

$$Z[B] = \delta(nB) \times e^{i \frac{nk}{4\pi} \int B \wedge B} .$$

( * If $n$ is a nontrivial divisor of $N$,

$e^{i \frac{nk}{4\pi} \int B \wedge B}$ itself is not gauge invariant.

However, we can check its gauge invariance if $\delta(nB) \neq 0$. )

For this vacuum, the deconfined (non-genuine) lines are generated by

$$W_{(0,n)} = W^n \quad \& \quad W_{(\frac{N}{n},k)} = H^{N/n} W^k .$$

# Our idea

We consider the $S^1$-compactified set up

$$M_3 \times S^1 \qquad \Big( \underbrace{\text{size}(M_3) \gg}_{\substack{\Downarrow \\ \text{We may consider} \\ \text{3d effective QFTs on } M_3}} \underbrace{\text{size}(S^1) \gg \frac{1}{\text{mass gap}}}_{\substack{\Downarrow \\ \text{No phase transition under compactification}}} \Big).$$

Under this compactification,

$$(\mathbb{Z}_N^{[1]})_{4d} \Longrightarrow (\mathbb{Z}_N^{[1]})_{3d} \times (\mathbb{Z}_N^{[0]})_{3d}.$$

$$B_{4d} = B_m + A \wedge \frac{d\tau}{L}.$$

We consider the gauging of $(\mathbb{Z}_N^{[0]})_{3d}$. (Temporal gauging)

$$Z_{tH}[\tilde{B}_e, B_m] = \int \mathcal{D}a \, e^{\frac{2\pi i}{N}\int_{3d} \tilde{B}_e \cup a} \, Z\left[B_m + a \wedge \frac{d\tau}{L}\right].$$

This 3d theory enjoys $\underset{\text{magnetic}}{\tilde{\mathbb{Z}}_N^{[1]}} \times \underset{\text{electric}}{\mathbb{Z}_N^{[1]}}$ symmetry!

So, we can apply the Wilson-'t Hooft scheme in a naive way.

# Positivity of $Z_{tH}[\tilde{B}_e, B_m]$

If 4d theory is unitary,

$$Z_{tH}[\tilde{B}_e, B_m] \geq 0.$$

(Proof) We consider the reflection map

$$\theta : M_3 \times S^1 \longrightarrow M_3 \times S^1, \quad (x, \tau) \mapsto (x, -\tau).$$

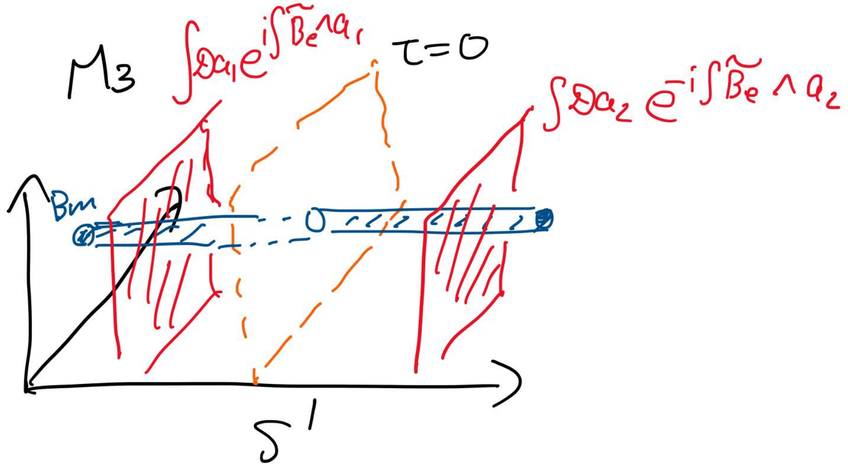
We note that

$$Z_{tH}[\tilde{B}_e, B_m] = \int \mathcal{D}a \, e^{i\int_{M_3} \tilde{B}_e \wedge a} \, Z[B_m + a \wedge \tfrac{d\tau}{L}]$$

$$= \int \mathcal{D}a_1 \mathcal{D}a_2 \, e^{i\int_{M_3} \tilde{B}_e \wedge (a_1 - a_2)} \, Z[B_m + (a_1 - a_2) \wedge \tfrac{d\tau}{L}].$$

$$\geq 0$$

by reflection positivity.



Classification as topological states $\overset{1:1}{\Longleftrightarrow}$ Wilson-'t Hooft classification

$$\boxed{\mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} \mathbb{Z}_{\underline{\underline{n}}}^{[1]} \text{ enriched with the level-}k\ \mathbb{Z}_{\underline{\underline{n}}}^{[1]}\text{-SPT}}$$

$\mathbb{Z}_{N/n}$ – TQFT

Proof

$$Z_{eff}[B_{4d}] = \frac{|H^0(M_4;\mathbb{Z}_{N/n})|}{|H^1(M_4;\mathbb{Z}_{N/n})|} \sum_{[b]\in H^2(M_4;\mathbb{Z}_{N/n})} \exp\left(\frac{2\pi i}{N/n}\int b\cup B_{4d}\right)$$

$$\times \exp\left(\frac{2\pi i k}{n}\int \frac{1}{2} P_2\left(\frac{B_{4d}}{N/n}\right)\right).$$

$\Downarrow$ temporal gauging

$B_{4d} = B_m + a\wedge \frac{d\tau}{L}$

$\int \mathcal{D}a \exp\left(\frac{2\pi i}{N}\int a\cup \tilde{B}_e\right) Z[B_m + a\frac{d\tau}{L}]$

$$Z_{tH} = N^{\beta_1(M_3)-1}\cdot \delta_N\left[n B_m, \frac{N}{n}\tilde{B}_e + k B_m\right]$$

$$\boxed{\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N \longrightarrow H = \left\{x(\underline{\underline{n}},0) + y\left(k, -\frac{N}{\underline{\underline{n}}}\right)\right\}}$$

(i.e. $W_{(0,n)}(C) = W^n(C)$ and $W_{(N/n,k)}(C) = H^{N/n}(C)\, W^k(C)$ are deconfined)

# Application to gapped vacua of $\mathcal{N}=1^*$ SYM

Donagi, Witten '96 : Gapped vacua of $\mathcal{N}=1^*$ SYM fit into the Wilson-'t Hooft classification.

$\mathcal{N}=1^*$ SYM = $\underbrace{\mathcal{N}=4 \text{ SYM}}$ + soft susy breaking to $\mathcal{N}=1$.

(= mass term to $\mathcal{N}=1$ chiral multiplets)

$A_\mu$, $\lambda$ — $\mathcal{N}=1$ vect. multiplet

$\lambda_1$, $X_1$ ($=\phi_1+i\phi_2$) — $\mathcal{N}=1$ chiral multiplet

$\lambda_2$, $X_2$ ($=\phi_3+i\phi_4$) — $\mathcal{N}=1$ chiral multiplet

$\lambda_3$, $X_3$ ($=\phi_5+i\phi_6$) — $\mathcal{N}=1$ chiral multiplet

w/ $W(X) = \frac{1}{g^2}\mathrm{tr}\left(X_1[X_2,X_3]\right) + \frac{m}{2g^2}\mathrm{tr}\left(X_1^2+X_2^2+X_3^2\right)$

- R-sym. of $\mathcal{N}=4$ SYM is $SU(4)$.

  $U(1)$ part is explicitly broken AT THE CLASSICAL LEVEL due to $W(X)$.

  $\Rightarrow$ $\mathcal{N}=1^*$ SYM does not have $\mathbb{Z}_{2N}$ axial symmetry unlike $\mathcal{N}=1$ SYM.

# Gapped vacua

<u>Classical vacua</u> $\frac{\partial W}{\partial X} = 0$

$$\Longleftrightarrow \quad [X_i, X_j] = -m\varepsilon_{ijk} X_k \quad : \text{ eq. for } \mathrm{Lie}(SU(2)) \text{ algebra}$$

Classical config. then takes the form of

$$X_i = m \begin{pmatrix} J_i^{(n_1)} & & & \\ & J_i^{(n_2)} & & \\ & & \ddots & \\ & & & J_i^{(n_\ell)} \end{pmatrix}$$

w/ $J_i^{(n)}$ : generator for n-dim rep. of $\mathrm{Lie}(SU(2))$ & $n_1 + \cdots + n_\ell = N$

In particular when

$$X_i = m \, \mathbb{1}_n \otimes J_i^{(N/n)},$$

the low-energy gauge group does not have $U(1)$'s :

$$SU(N) \xrightarrow{\text{Higgsing}} \frac{SU(n) \times \mathbb{Z}_N}{\mathbb{Z}_n}.$$

[cf. Donagi, Witten '96, Dorey '99]

As $\mathcal{N}=1$ $SU(n)$ SYM is confining,

the deconfined electric line is $W^n$ and the theory flows to $\mathbb{Z}_{N/n}$ TQFT.

# Formal S & T operations

We can apply formal S & T operations in the space of 4d QFTs with $\mathbb{Z}_N^{[1]}$ sym:

$$\begin{cases} S: Z[B_{4d}] \mapsto S\cdot Z[B_{4d}] = \int \mathcal{D}b_{4d}\, Z[b_{4d}]\, e^{\frac{2\pi i}{N}\int b_{4d}\cup B_{4d}} \\ T: Z[B_{4d}] \mapsto T\cdot Z[B_{4d}] = Z[B_{4d}]\, e^{\frac{2\pi i}{N}\int \frac{1}{2}P_2(B_{4d})} \end{cases}$$

Correspondingly,

electro-magnetic dual

$$S\cdot Z_{tH}\left[\begin{pmatrix}\tilde{B}_e\\ B_m\end{pmatrix}\right] = Z_{tH}\left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}\tilde{B}_e\\ B_m\end{pmatrix}\right]$$

Witten effect

$$T\cdot Z_{tH}\left[\begin{pmatrix}\tilde{B}_e\\ B_m\end{pmatrix}\right] = Z_{tH}\left[\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}\tilde{B}_e\\ B_m\end{pmatrix}\right]$$

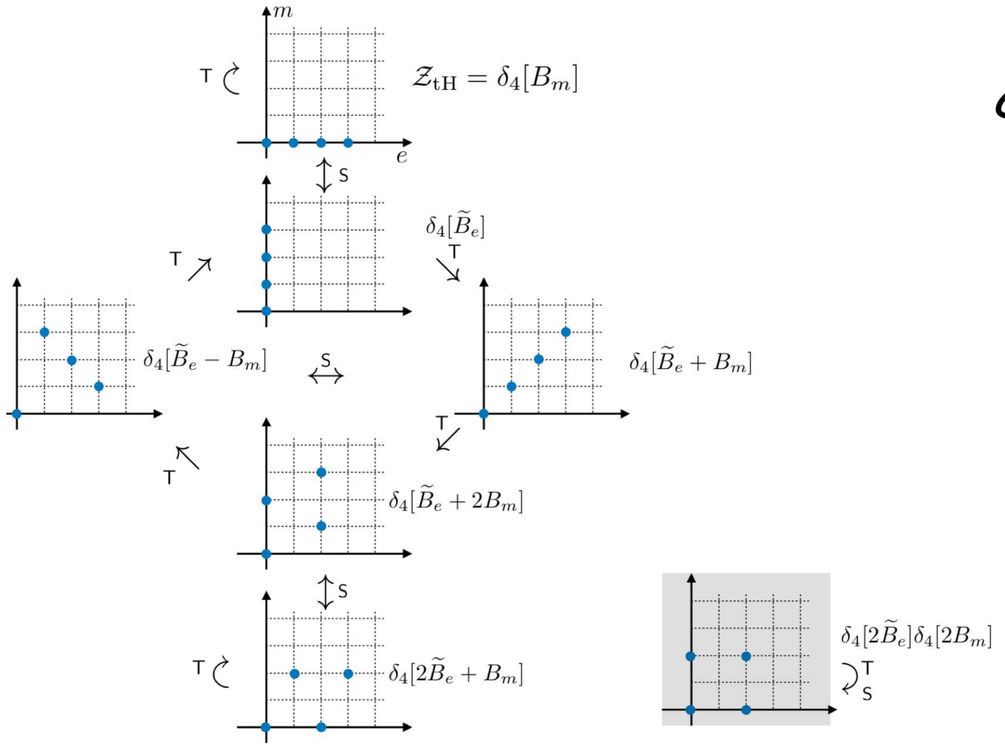
These operations form an $SL(2,\mathbb{Z}_N)$ automorphism on $\tilde{\mathbb{Z}}_N\times\mathbb{Z}_N$, and we can relate different gapped states specified by order-$N$ subgroups $H_1, H_2$ when they induce an isomorphism $H_1 \xrightarrow{\sim} H_2$.

ex. SU(4) case

When $N_c=4$, the pattern of classical vacua is

| Higgs vev | $SU(4) \xrightarrow{\text{Higgs}}$ | $(\mathbb{Z}_4^{[1]})_{4\text{d}} \xrightarrow{\text{SSB}}$ | $\mathcal{Z}_{\text{tH}}[\widetilde{B}_e, B_m]$ |
|---|---|---|---|
| $J_i^{(d=4)}$ | $\mathbb{Z}_4$ | $1$ | $\delta_4[B_m]$ |
| $\mathbf{1}_2 \otimes J_i^{(d=2)}$ | $[SU(2)\times\mathbb{Z}_4]/\mathbb{Z}_2$ | $(\mathbb{Z}_2^{[1]})_{4\text{d}}$ | $\delta_4[2B_m, 2\widetilde{B}_e + kB_m]$ |
| $\mathbf{1}_4 \otimes J_i^{(d=1)} = 0$ | $SU(4)$ | $(\mathbb{Z}_4^{[1]})_{4\text{d}}$ | $\delta_4[\widetilde{B}_e + kB_m]$ |

We note that $\mathcal{N}=1$ SU(n) SYM has n vacua, and its label $k=0,1,\cdots,n-1$ appears in the low-energy partition function.




← These partition functions are consistent w/ $SL(2,\mathbb{Z}_N)$ actions.

Direct computation of $\langle W_{(m,n)}(C,\Sigma) \rangle$

We can also directly check the behaviors of loop operators in each gapped vacua.

(work in progress w/ Nguyen, Ünsal)

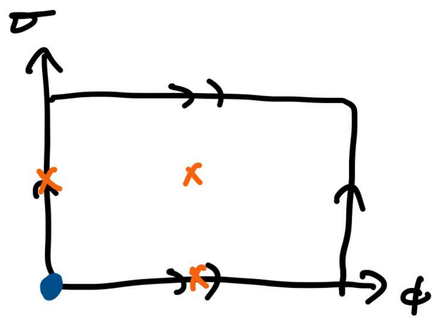
For this purpose, we consider $M_4 = \mathbb{R}^3 \times S^1$.

In this setup,

$$A_\mu \text{ in 4d} \longrightarrow \begin{cases} \phi : \text{holonomy} \\ \sigma : \text{dual photon} \end{cases}$$

and Dorey '99 proposed its nonperturbative superpotential (elliptic superpotential)

<u>SU(2) case</u>

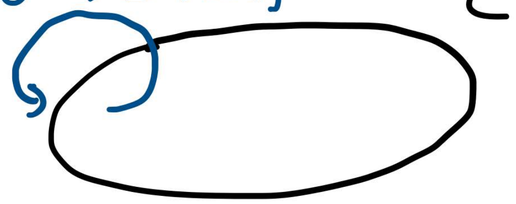



× : $\frac{\partial W}{\partial(\sigma+i\phi)} = 0$

• : pole singularity

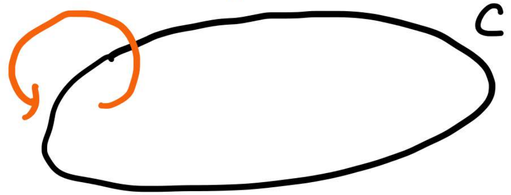
Wilson loop $W(C)$ :

$\sigma \to \sigma + 2\pi\mu$

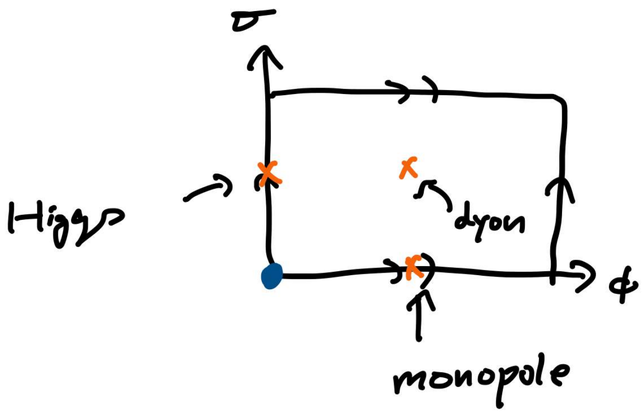



't Hooft loop $H(C,\Sigma)$

$\phi \to \phi + 2\pi\mu$



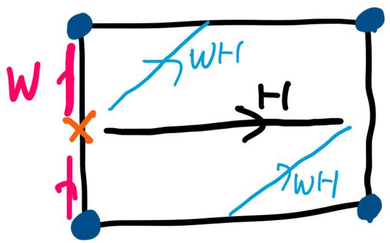

SU(2) case




$\times$ : $\frac{\partial W}{\partial(\sigma+i\phi)} = 0$

$\bullet$ : pole singularity

Then, let's take $(\phi,\sigma) = (0,\pi)$ vacuum ($\longleftrightarrow$ Higgs vacuum)




For $W$, the trajectory hits the singularity and it bounces back.

$\rightsquigarrow \langle W(C)\rangle$ : perimeter law.

while $\langle H\rangle$, $\langle HW\rangle$ : area law.

In this way, we can explicitly check the consistency of the behaviors of the loop op.

# Summary

- We relate two classification schemes for gapped phases of 4d QFTs w/ $\mathbb{Z}_N^{[1]}$ sym.

$$\mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} \mathbb{Z}_{\underline{\underline{n}}}^{[1]} \quad \text{enriched with the level-} k \ \mathbb{Z}_{\underline{\underline{n}}}^{[1]}\text{-SPT}$$

$$\overset{1:1}{\Longleftrightarrow} \quad \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N \longrightarrow H = \{x(\underline{\underline{n}},0) + y(k, -\tfrac{N}{\underline{\underline{n}}})\} \quad \text{in Wilson-'t Hooft classification.}$$

$$\boxed{\begin{aligned} Z_{\text{eff}}[B_{4d}] &= \frac{|H^0(M_4;\mathbb{Z}_{N/n})|}{|H^1(M_4;\mathbb{Z}_{N/n})|} \sum_{[b]\in H^2(M_4;\mathbb{Z}_{N/n})} \exp\left(\frac{2\pi i}{N/n}\int b\cup B_{4d}\right) \\ &\quad \times \exp\left(\frac{2\pi i k}{n}\int \frac{1}{2} P_2\left(\frac{B_{4d}}{N/n}\right)\right). \\ &\Big\Downarrow \ \text{temporal gauging} \\ &\quad B_{4d} = B_m + a\wedge \frac{d\tau}{L} \\ &\quad \int \mathcal{D}a \exp\left(\frac{2\pi i}{N}\int a\cup \tilde{B}_e\right) Z[B_m + a\tfrac{d\tau}{L}] \\ Z_{tH} &= N^{\beta_1(M_3)-1}\cdot \delta_N\left[n B_m, \frac{N}{n}\tilde{B}_e + k B_m\right] \end{aligned}}$$

- We consider the formal S & T operations in the space of 4d QFTs.

- We apply these results to 4d $\mathcal{N}=1^*$ SYM theory.